

Stationary state entanglement and total correlation of two qubits or qutrits

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We investigate the mutual information and entanglement of stationary state of two locally driven qubits under the influence of collective dephasing. It is shown that both the mutual information and the entanglement of two qubits in the stationary state exhibit damped oscillation with the scaled action time γT of the local external driving field. It means that we can control both the entanglement and total correlation of the stationary state of two qubits by adjusting the action time of the driving field. We also consider the influence of collective dephasing on entanglement of two qutrits and obtain the sufficient condition that the stationary state is entangled.

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I. INTRODUCTION

Quantum entanglement plays a very important role in quantum information processes, which can exhibit the nature of a nonclassical correlation between quantum systems that have no classical interpretation [1]. Entanglement of two or more subsystems can be destroyed by the interaction between quantum systems of interest and its surrounding environments. Certain kind of the interaction between the physical system and environments can lead to the collective dephasing, which occurs in various physical systems such as two coupled quantum dots. Recently, the quantum information processes in the presence of the collective dephasing have intrigued much attention [2, 3, 6, 7]. Khodjasteh and Lidar have investigated the universal fault-tolerant quantum computation in the presence of spontaneous emission and collective dephasing [2]. Hill and Goan have studied the effect of dephasing on proposed quantum gates for the solid-state Kane quantum computing architecture [3]. Collective dephasing allows the existence of the so-called decoherence-free subspace [4]. It has been shown that some kinds of entangled states of two qubits are very fragile in the presence of collective dephasing while the others are very robust [6]. So, it is very desirable to protect the fragile entangled states from completely losing their entanglement in the collective dephasing.

The mutual information is also a very important quantity which has been used to measure the total correlation of two subsystems. In this paper, we comparatively investigate the mutual information measuring the total correlation and the stationary state entanglement quantified by concurrence of two locally driven qubits in the presence of collective dephasing. It is shown that one can transform the fragile entangled states into the stationary entangled states under the collective dephasing by mak-

ing use of a finite-time external driving field. We further find that the stationary state achieving the local maximal value of concurrence also have the local maximal value of mutual information. Moreover, we investigate the model of two spin 1 in the presence of collective dephasing and obtain the general sufficient condition that the stationary state is entangled.

The disentanglement of entangled states of qubits is a very important issue for quantum information processes, such as the solid state quantum computation. For example, in quantum registers, some kinds of undesirable entanglement between the qubits can lead to the decoherence of the qubit [5]. Recently, Yu and Eberly have found that the time for decay of the qubit entanglement can be significantly shorter than the time for local dephasing of the individual qubits [6, 7]. The collective dephasing can be described by the master equation

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{\gamma}{2}(2\hat{J}_z\hat{\rho}\hat{J}_z - \hat{J}_z^2\hat{\rho} - \hat{\rho}\hat{J}_z^2), \quad (1)$$

where γ is the decay rate. \hat{J}_z are the collective spin operator defined by

$$\hat{J}_z = \sum_{i=1}^2 \hat{\sigma}_z^{(i)}/2, \quad (2)$$

where $\hat{\sigma}_z$ for each qubit is defined by $\hat{\sigma}_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. Previous studies have shown that two of the four Bell states $|\Psi^\pm\rangle \equiv \frac{\sqrt{2}}{2}(|11\rangle \pm |00\rangle)$ are fragile states in the collective dephasing channel, while the others Bell states $|\Phi^\pm\rangle \equiv \frac{\sqrt{2}}{2}(|10\rangle \pm |01\rangle)$ are robust entangled states.

Meanwhile, much attention has been paid to the quantum information processes in which the basic element is a qutrit. Quantum communication complexity protocol with two entangled qutrits has been proposed [8], and it has been proven that, for a broad class of protocols the entangled state of two qutrits can enhance the efficiency of solving the problem in the quantum protocol over any classical one if and only if the state

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violates Bell's inequality for two qutrits. By making use of the entangled qutrits, a generalization of Ekert's entanglement-based quantum cryptographic protocol has been discussed [9]. By means of the orbital angular momentum of the photons, the experimental implementation of producing, transmitting, and reconstructing the qutrit has been reported [10]. The generation, manipulation and measurement of entangled qutrits have also been experimentally studied by utilizing spontaneous parametric down converted photons and unbalanced 3-arm fiber optic interferometers [11]. In ideal situations, entangled qutrits provide better security than qubits in quantum bit commitment. However, it has been shown that qutrits with even a small amount of decoherence cannot offer increased security over qubits [12]. It motivates us to investigate how the decoherence can affect the entanglement of two qutrits. We consider two qutrits in collective dephasing channel and obtain the sufficient condition for the entangled stationary state. By making use of this sufficient condition, we can distinguish the robust entangled state of two qutrits from the fragile entangled state.

This paper is organized as follows: In section II, we comparatively investigate the mutual information measuring the total correlation and the stationary state entanglement quantified by concurrence of two locally driven qubits in the presence of collective dephasing. In section III, we consider two qutrits in collective dephasing channel and give a sufficient criterion that a class of initial entangled state of two qutrits which is robust against the collective dephasing. In section IV, there are some conclusions.

II. THE ENTANGLEMENT AND MUTUAL INFORMATION OF THE STATIONARY STATE

In this section, we investigate the model in which two qubits are exposed in a collective dephasing environment and one of two qubits is simultaneously driven by a finite-time external field. The dynamics of two qubits can be described by the following master equation

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{2}[\Omega_1(t)\hat{\sigma}_x^{(1)}, \hat{\rho}] + \frac{\gamma}{2}(2\hat{J}_z\hat{\rho}\hat{J}_z - \hat{J}_z^2\hat{\rho} - \hat{\rho}\hat{J}_z^2), \quad (3)$$

where $\Omega_1(t) = \Omega_1\Theta(T-t)$ is the intensity of the time-dependent external driving field acted on the qubit 1, and $\Theta(x)$ is the unit step function and equals one for $x \geq 0$ and equals zero for $x < 0$. In the following calculations, we will show that the action time T of the external driving field plays a special role in the stationary state entanglement and the total correlation of two qubits. After numerically solving the master equation (3), we can obtain the stationary state density matrix of two qubits. Without loss of generality, the stationary state density matrix has the form

$$\rho_s = a(\frac{\Omega_1}{\gamma}, \gamma T)|11\rangle\langle 11| + b(\frac{\Omega_1}{\gamma}, \gamma T)|10\rangle\langle 10|$$

$$+ c(\frac{\Omega_1}{\gamma}, \gamma T)|01\rangle\langle 01| + d(\frac{\Omega_1}{\gamma}, \gamma T)|00\rangle\langle 00| \\ + f(\frac{\Omega_1}{\gamma}, \gamma T)|10\rangle\langle 01| + f^*(\frac{\Omega_1}{\gamma}, \gamma T)|01\rangle\langle 10|. \quad (4)$$

In order to quantify the degree of entanglement, we adopt the concurrence C defined by Wootters [13]. The concurrence varies from $C = 0$ for an unentangled state to $C = 1$ for a maximally entangled state. For two qubits, in the "Standard" eigenbasis: $|1\rangle \equiv |11\rangle$, $|2\rangle \equiv |10\rangle$, $|3\rangle \equiv |01\rangle$, $|4\rangle \equiv |00\rangle$, the concurrence may be calculated explicitly from the following:

$$C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \quad (5)$$

where the $\lambda_i (i = 1, 2, 3, 4)$ are the square roots of the eigenvalues in decreasing order of magnitude of the "spin-flipped" density matrix operator $R = \rho_s(\sigma^y \otimes \sigma^y)\rho_s^*(\sigma^y \otimes \sigma^y)$, where the asterisk indicates complex conjugation. The concurrence related to the density matrix ρ_s can be written as

$$C_s = 2 \max[0, |f(\frac{\Omega_1}{\gamma}, \gamma T)| - \sqrt{a(\frac{\Omega_1}{\gamma}, \gamma T)d(\frac{\Omega_1}{\gamma}, \gamma T)}]. \quad (6)$$

The mutual information of the stationary state of two qubits is defined by $I(\rho_s) = S(\rho_1) + S(\rho_2) - S(\rho_s)$, where $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the Von-Neumann entropy of ρ , and $\rho_1 = \text{Tr}_2 \rho_s$ and $\rho_2 = \text{Tr}_1 \rho_s$ are the reduced density operators of the qubit 1 and the qubit 2 respectively. The mutual information can be used to measure the total correlation of two qubits. We can easily obtain the analytical expression of the mutual information I as follows,

$$I = -(a+b)\log_2(a+b) - (c+d)\log_2(c+d) \\ - (a+c)\log_2(a+c) - (b+d)\log_2(b+d) \\ + a\log_2 a + d\log_2 d \\ + \beta_+\log_2 \beta_+ + \beta_-\log_2 \beta_-, \quad (7)$$

where

$$\beta_{\pm} = \frac{b+c \pm \sqrt{(b-c)^2 + 4|f|^2}}{2}. \quad (8)$$

Firstly, we consider the case in which two qubits are initially in the Bell state $|\Phi^-\rangle$, i.e., the Bell singlet state. In Fig.1, the stationary state concurrence C_s and the mutual information I of the stationary state are plotted as the function of the parameter γT . It is shown that the values γT_i^m ($i = 0, 1, 2, \dots, 9$ in this case) of the scaled action time which locally maximize the stationary state entanglement always locally maximize the mutual information, i.e. the total correlation of two qubits. Both the mutual information of the stationary state and the stationary state concurrence oscillate with γT , which implies one can control both the entanglement and mutual information of the stationary state of two qubits in the collective dephasing environment by adjusting the scaled

action time γT of the locally driving field. For analyzing Fig.1 in details, we orderly label those special points with γT_i ($i = 1, 2, 3, \dots$) at which the stationary states transit from the entangled state to the separable state or vice versa. Namely, $C_s(\gamma T) > 0$ when $0 \leq T < T_1$ or $T_{2k} < T < T_{2k+1}$ ($k = 1, 2, \dots, 9$ in Fig.1), and $C_s(\gamma T) = 0$ when $T_{2k-1} \leq T \leq T_{2k}$ ($k = 1, 2, \dots, 9$ in Fig.1) or $T \geq T_{19}$. In those windows of $T \in [T_{2k-1}, T_{2k}]$, the initial bell singlet state eventually becomes separable in the dynamical evolution governed by the Eq.(3) though the total correlation of two qubits in the stationary state is not zero. The length $\gamma(T_{2k} - T_{2k-1})$ ($k = 1, 2, \dots, 9$) of those windows increase with k . We can find that in the case with strong intensity of external field, i.e. $\Omega_1 \gg \gamma$, the stationary state concurrence decreases from one to zero with the increase of the scaled action time γT from zero to γT_1 . With the further increase of γT from γT_2 to γT_1^m , the stationary state concurrence increases from zero to the local maximal value $C_s(\gamma T_1^m)$. The local maximal values $C_s(\gamma T_i^m)$ are in decreasing order of magnitude, namely $C_s(\gamma T_i^m) > C_s(\gamma T_{i+1}^m)$. Now we turn to discuss the mutual information of the stationary state of two qubits. $I(\gamma T_i^m)$ are also the local maximal values. When $T_{2k} \leq T \leq T_{2k+1}$, the total correlation of two qubits firstly increases with γT and achieves its local maximum at γT_k^m , then decreases with γT . This dynamical behavior is consistent with the entanglement of two qubits. When $T_{2k-1} \leq T \leq T_{2k}$, the stationary state is separable but the total correlation is not zero and has a peak value.

In what follows, we consider the case in which two qubits are initially in the fragile entangled state $|\Psi^+\rangle$. If the external driving field is absence, two qubits lose its entanglement more rapidly than the disappearance of its coherence. Certainly, it is a disadvantage for many quantum information processes such as the solid state quantum computation. Here, we shall show that a finite-time external driving field can protect two qubits from completely disentanglement. In Fig.2, we display the stationary state concurrence and the mutual information of the stationary state as the function of the parameter γT . When γT is chosen as a very small value, the stationary state is still separable and the total correlation is weaken. For conveniently analyzing the Fig.2, we still orderly label those special points with $\gamma T'_i$ ($i = 1, 2, 3, \dots$) at which the stationary states transit from the entangled state to the separable state or vice versa. After carefully comparing the Fig.2(a) and Fig.1(a), we may observe

$$T'_i \geq T_i, \quad \text{and} \quad T'_i \leq T_i. \quad (9)$$

The above inequalities imply that two qubits initially in Bell states $|\Phi^-\rangle$ or $|\Psi^+\rangle$ can not simultaneously evolve into stationary entangled states in their evolutions governed by Eq.(3) with fixed parameter T . Being similar with Fig.1, we can also find that in the range of $T'_{2k-1} \leq T \leq T'_{2k}$, the behaviors of entanglement and total correlation are consistent. Both the concurrence and mutual information simultaneously achieve their local maximal values.

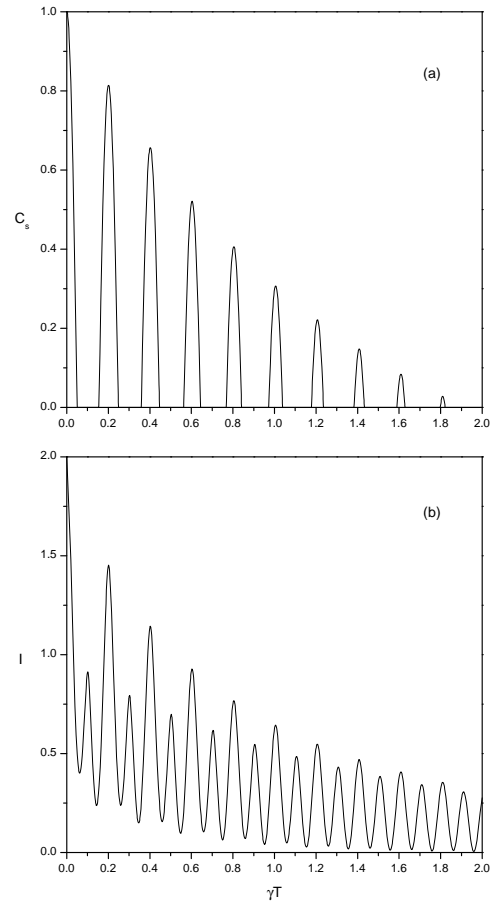


FIG. 1: The stationary state concurrence C_s and the mutual information I of the stationary state are plotted as the function of the parameter γT with $\Omega_1/\gamma = 31.25$. In this case, two qubits are initially in the Bell state $|\Phi^-\rangle$. By comparing the concurrence in (a) and the mutual information in (b), it is not difficult to verify that the mutual information is always larger than the concurrence.

III. TWO QUTRITS IN THE COLLECTIVE DEPHASING CHANNEL

In this section, we briefly discuss the possible generalization to two spin 1 $\hat{j}(1)$ and $\hat{j}(2)$, i.e. two qutrits. $\{|-1\rangle, |0\rangle, |1\rangle\}$ are three eigenvectors of the z -component of a spin 1. The collective dephasing of two spin 1 can be described by the master equation

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{\gamma}{2} (2\hat{L}_z \hat{\rho} \hat{L}_z - \hat{L}_z^2 \hat{\rho} - \hat{\rho} \hat{L}_z^2), \quad (10)$$

where γ is the decay rate. \hat{L}_z are the collective spin operator defined by

$$\hat{L}_z = \sum_{i=1}^2 \hat{j}_z^{(i)}, \quad (11)$$

where \hat{j}_z for each spin 1 is defined by $\hat{j}_z = |1\rangle\langle 1| - |-1\rangle\langle -1|$. By making use of the well-known Peres-Horodecki criterion [14, 15], we can investigate whether

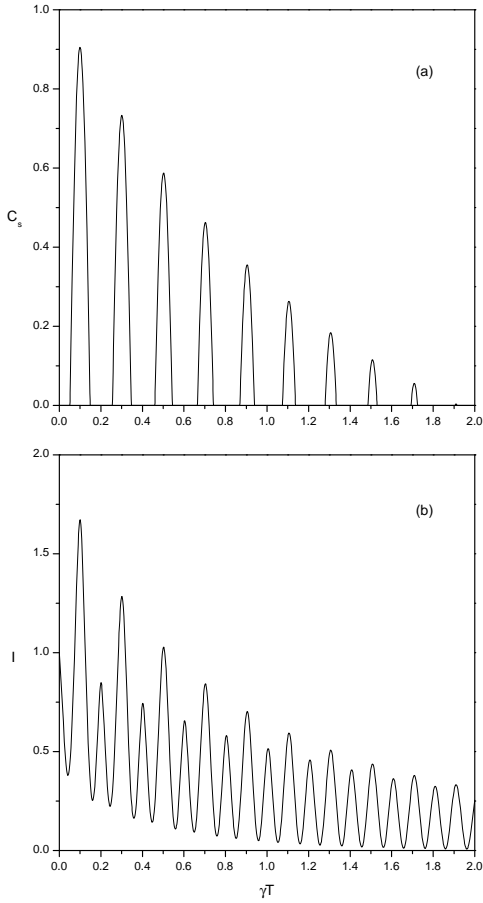


FIG. 2: The stationary state concurrence C_s and the mutual information I of the stationary state are plotted as the function of the parameter γT with $\Omega_1/\gamma = 31.25$. In this case, two qubits are initially in the Bell state $|\Psi^+\rangle$.

the stationary state is entangled or not. For any initial density operator ρ , the sufficient condition that the stationary state is entangled can be derived as follows: The equation

$$x^3 - \xi x^2 + \zeta x + \eta = 0, \quad (12)$$

where

$$\begin{aligned} \xi &= \rho_{1,1;1,1}^2 + \rho_{0,0;0,0}^2 + \rho_{-1,-1;-1,-1}^2, \\ \zeta &= \rho_{1,1;1,1}\rho_{0,0;0,0} + \rho_{1,1;1,1}\rho_{-1,-1;-1,-1} \\ &\quad + \rho_{0,0;0,0}\rho_{-1,-1;-1,-1} - |\rho_{0,-1;-1,0}|^2 \\ &\quad - |\rho_{1,0;0,1}|^2 - |\rho_{1,-1;-1,1}|^2, \\ \eta &= -\rho_{1,1;1,1}\rho_{0,0;0,0}\rho_{-1,-1;-1,-1} \end{aligned}$$

$$\begin{aligned} &+ \rho_{1,1;1,1}|\rho_{0,-1;-1,0}|^2 + \rho_{-1,-1;-1,-1}|\rho_{1,0;0,1}|^2 \\ &+ \rho_{0,0;0,0}|\rho_{1,-1;-1,1}|^2 \\ &- \rho_{1,-1;-1,1}\rho_{0,-1;-1,0}^* \rho_{1,0;0,1}^* \\ &- \rho_{1,-1;-1,1}^* \rho_{0,-1;-1,0} \rho_{1,0;0,1}, \\ &\rho_{i,j;k,l} \equiv \langle i, j | \rho | k, l \rangle, \end{aligned} \quad (13)$$

has a negative root, or at least one of the following two inequalities are satisfied

$$|\rho_{1,-1;0,0}|^2 > \rho_{1,0;1,0}\rho_{0,-1;0,-1}, \quad (14)$$

or

$$|\rho_{0,0;-1,1}|^2 > \rho_{0,1;0,1}\rho_{-1,0;-1,0}. \quad (15)$$

IV. CONCLUSIONS

In this paper, we comparatively investigate the mutual information of the stationary state and the stationary state entanglement quantified by concurrence of two locally driven qubits in the presence of collective dephasing. We show that one can transform the fragile entangled states into the stationary entangled states under the collective dephasing by making use of a finite-time external driving field. The local maximal value of the stationary state concurrence corresponds to the local maximal value of mutual information, i.e. the total correlation of two qubits. We show how a finite-time external driving field can control the entanglement and total correlation of the stationary state of two qubits under the collective dephasing environment. Moreover, we investigate the model of two spin 1 in the presence of collective dephasing. Since the stationary state strongly depends its initial state in this case, we obtain the general sufficient condition that the stationary state is entangled. It is very interesting to study how to protect the fragile entangled state of two qubit in the collective dephasing environment. The results will be discussed elsewhere. In the future work, it may be very interesting to apply the present results to some realistic quantum information processes, such as quantum computation based on the quantum dots or Josephson Junctions.

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